Oct 1, 1982

Dear Stephanie,

In the first part of this letter — mailed 9 days ago — I stopped in the middle of emphasizing the big role of differential equations in formulating physical laws. The example I chose was that of the so called "simple harmonic motion" of a weight hanging from a spring. A somewhat simpler and earlier example is the theory of falling bodies and projectiles of Galileo. Of course Galileo did not use differential equations and did not have to because in his case the physical law can be stated and analyzed using less sophisticated tools. However it is interesting to see that his whole theory is a consequence of the differential equations $\frac{d^2y}{dt^2} = -32$, $\frac{d^2x}{dt^2} = 0$, where $x(t)$ and $y(t)$ are the x and y coordinates of a freely moving body in space as a function of the time. In this case the differential equation can be solved by ordinary integration. The first reads $\frac{d}{dt}(\frac{dy}{dt}) = -32$. Hence $\frac{dy}{dt} = -32t + C_1$, where C_1 is an unknown constant. Integrating again we find that $y(t) = -16t^2 + C_1t + C_2$ where C_2 is another unknown constant. Similarly we find that $x(t) = C_3t + C_4t$. Thus once we know the four constants C_1, C_2, C_3, C_4 we know the whole trajectory of our body. These however,can be determined from the position of the body at $t = 0$. Let $x_0 = x(0), y_0 = y(0), v_0^x = \frac{dx}{dt}(0), v_0^y = \frac{dy}{dt}(0)$. Then letting $t = 0$ in the formula for $x(t)$ we get $x_0 = C_4$, and letting $t = 0$ in the formula for $y(t)$ we get $y_0 = C_2$. Similarly the formula $x'(t) = C_3$ gives us $v_0^x = C_3$ and the formula $y'(t) = -32t + C_1$ gives us $v_0^y = C_1$. In other words

$$x(t) = tv_0^x + x_0$$
$$y(t) = -16t^2 + v_0^y t + y_0$$

and we have explicit formula for the entire motion once we know the position and velocity components at $t = 0$. This is the sense in which the whole thing is contained entirely in the differential equations $\frac{d^2y}{dt^2} = -32$, $\frac{d^2x}{dt^2} = 0$.

Newton was concerned with the motion of a planet about the sun and (assuming the sun to be fixed at $(0, 0, 0)$) his fundamental differential equations for the coordinates $x(t), y(t), z(t)$ of a planet are

$$\frac{d^2x}{dt^2} = -MG\frac{x}{(\sqrt{x^2+y^2+z^2})^3}$$

$$\frac{d^2y}{dt^2} = -MG\frac{y}{(\sqrt{x^2+y^2+z^2})^3}$$

$$\frac{d^2z}{dt^2} = -MG\frac{z}{(\sqrt{x^2+y^2+z^2})^3}$$

where G and M are certain constants. These are much harder than those considered above but Newton managed to deal with them and find a general formula for $x(t), y(t), z(t)$ with given initial position and velocity. In particular he was able to show that whenever the speeds are not too great then the planet moves in an ellipse with the sun at one focus. The argument is a bit lengthy and I won't give it but it is well within your capacity to understand. Maybe they gave it to you in physics last year. Newton went further though and wrote down the differential equations of motion of the whole solar system which results from the attraction of the planets for each other as well as for the sun. In this case we need $3n+3$ equations and there are $3n+3$ functions to determine $x_1(t), y_1(t), z_1(t), \ldots, x_n(t), y_n(t), z_n(t), x(t), y(t), z(t)$. Here $y_j(t)$ is the y coordinate of the j-th planet, and $x(t)$ is the x coordinate of the sun and similarly for the others. A typical one of the $3n+3$ equations is

$$m_j\frac{d^2x_j}{dt^2} = -\sum_{i=1, i\neq j}^{n} \frac{Gm_im_j(x_j-x_i)}{[(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2]^{3/2}}$$

$$-\frac{GMm_j}{[(x_i-x)^2+(y_i-y)^2+(z_i-z)^2]^{3/2}}$$

These equations are very hard to solve and no one has ever done it — even in the case $n=2$ so that there are 3 bodies — the sun and two planets (or the sun, earth and moon). Nevertheless the differential equations completely determine the motion (once the initial conditions are given) and one can learn a great deal about the motion by studying the differential equation and proving theorems about their solutions even though one cannot find explicit formulae.

So much for the seventeenth century. What happened in the eighteenth? A very large part falls under one of the following three headings

1. Studies of planetary motion — especially the sun, earth and moon (the so called linear problem) using Newton's differential equations.

2. Extension of Newton's laws to continuous matter as opposed to "particles" of matter — especially the laws of fluid motion.

3. Developments in number theory including the proofs of many of Fermat's statements.

The chief mathematicians of the century were Euler (1707-1783), D'Alambert (1717-1783), Bernoulli (1700-1782), Lagrange (1736-1812) and Laplace (1749-1827). Euler and Lagrange worked on all three subjects. The others confined themselves to (1) and (2).

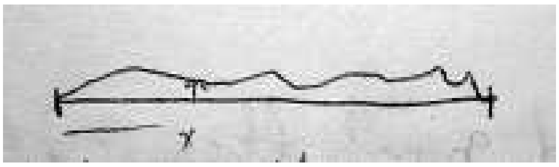
Oct 2, 1982

I will say no more about (1) at this point, except to say that many difficult theorems were proved and that finding such continues to the present day. It was an important advance when G. W. Hill (1838-1914) pointed out the importance of spending less time seeking explicit solutions and more time studying the qualitative properties of the solutions by abstract methods. This idea greatly influenced the work of Henri Poincare (1854-1912), one of the great mathematicians of all times and Poincare in turn influenced G. D. Birkhoff (1884-1944) who devoted much of his very distinguished career to furthering the work of Poincare on the subject. Birkhoff's work led to the new subject of ergodic theory in 1931 which also has important connection with probability. From an abstract point of view ergodic theory is a sort of mixture of group theory and measure theory but I should not try to explain further until you know what group theory and measure theory are. It is one of my main interests and perhaps the chief interest of Professor Kakutani at Yale.

Oct 5, 1982

Among the chief followers of G. D. Birkhoff in celestial mechanics are A. N. Kolmogorov and V. I. Arnold of Russia, J. Moser of Zurich (for many years at the Courant Institute in New York) and Stephen Smale of Berkley. Another who died recently is C. L. Siegel at Göttingen who was the teacher of Moser.

Getting back to the 18th century let us look at (2). In planetary motion one thinks of the planets as points and is interested only in their position. In so called continuum mechanics one takes a piece of matter and studies the relative motion of its parts. In dealing with liquids and gasses one speaks of fluid mechanics and in dealing with solids one speaks of elasticity and plasticity. The main 18th century progress was in formulating the laws of fluid mechanics and here an important new tool was introduced — the notion of a *partial differential equation.* This is just what you might guess; an equation asserting certain relations between the partial derivatives of one or more functions of several variables. Consider for example the motion of a stretched plucked string as on a violin. At any given moment of time the displacement of the string from its rest position will be described by a function f of one variable which at the distance x from one end specifies the displacement $f(x)$ at that distance.



Since this changes with time the displacement is a function $f(x,t)$ of the *two* variables x and t. The physics of vibrating string (as long as the vibrations are small) is completely described by the partial differential equation $\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$ where v is a constant depending on the density and stretchability of the string. Shortly after Ann dropped Math last year she encouraged me to teach her some by letter and I wrote several letters (at least two) showing how what she had just learned could be used to make sense of the partial differential equations of physics, including this one. Ask her to show them to you if you want really to see how solving such equations is vital for physics. At any rate the study of partial differential equations soon became a very important part of mathematics and continues so to the present day. Little progress was made until the early nineteenth century when "harmonic analysis" came to the rescue — more of this below.

The actual partial differential equations of fluid mechanics are a little complicated and I won't write them down for you. (Note that a string can be considered as a one dimensional fluid). Instead let me pass to 3). What happened under 3) was first that many of Fermat's statements were proved by Euler and Lagrange and more importantly that a systematic theory began to emerge. Lagrange in particular solved the following problems. Given arbitrary whole numbers A, B, C, D, E, F find:
a) *all* pairs x, y of *rational* numbers such that

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

b)*all* pairs x, y of integers x, y such that

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Of course there may be none at all and part of the problem is to determine in which cases there are no solutions. Although these problems are easy to state and their solutions are beautiful they are not easy to find and I do *not* suggest that you try to find them. However you should be able to do the first step in a) which is to show that by a simple change of variables like $x = X + \alpha$, $y = Y + \beta$ or $x = \frac{\alpha X+\beta}{\gamma X+\delta}$ and $y = \frac{\alpha Y+\beta}{\gamma Y+\delta}$ problem a) can be reduced to one of the following two special cases:

1. $x^2 - By^2 = C$
2. (2) $xy = C$

Of course (2) is easily solved but (1) is still difficult. You also might find it amusing to try to prove a very important and original discovery made by Euler; namely that for any $s > 1$ the sum of the infinite series

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots$$

can be factored as the infinite product

$$\left(\frac{1}{1-\frac{1}{2^s}}\right)\left(\frac{1}{1-\frac{1}{3^s}}\right)\left(\frac{1}{1-\frac{1}{5^s}}\right)\cdots$$

where the numbers that appear in the product are the *primes* 2, 3, 5, 7, 11, 13, 17, 19, 23 etc. Hint: first use the theory of geometric series to replace

$$\frac{1}{1-\frac{1}{p^s}} \text{ by } 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots$$

This is quite easy — at least on a formal level. Somewhat harder but perhaps within your powers is the following deduction that Euler made from his identity

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \ldots = \sum_{p-prime} \frac{1}{p}$$

diverges. You probably know the proof that $\sum_{n=1}^{n=\infty} \frac{1}{n}$ diverges. The problem is to show that the series still diverges if you leave out everything but the primes — by using Euler's identity to relate the two sums.

Many of the things that Euler and Lagrange discovered about number theory make more sense when you bring group theory into the picture. Indeed all the basic concepts of modern algebra — groups, rings and fields may be and largely were motivated by the needs of number theory. Moreover Lagrange made the first step in this direction in a paper written around 1770. In order to proceed much further with my story I need to talk about groups so let me tell you about them. Maybe you have already studied them but just in case I will start from the beginning.

Let G be any set. Let there be given a rule for "multiplying" two members of G together to get another member. We suppose that this rule satisfies the following three axioms.

1. $x(yz) = (xy)z$ for all x, y and z in G.
2. There is an element e in G (called the identity), such that $ex = xe$ for all x in G.
3. For each x in G there is a y in G such that $xy = yx = e$. (y is called the inverse of x).

We remark that it is easy to prove (try it) that e is unique — there cannot be two identities and also that x can have only one inverse. The inverse of x is usually denoted by x^{-1}. Note also that we do *not* assume that $xy = yx$. By definition a group is any set G together with a "multiplication" satisfying (1),(2),(3). In some groups it is true that $xy = yx$ for all x and y. These are called *commutative groups*.

Example 1 - G is the set of all real numbers and the "multiplication" is addition. In this case $e = 0$ and $x^{-1} = -x$. This group is commutative.

Example 2 - G is the set of all *positive* real numbers and the "multiplication" is the ordinary multiplication. Here $e = 1$ and $x^{-1} = \frac{1}{x}$. This group also is commutative.

Example 3 G is the group of all 2×2 matrices with real coefficients and determinant $\neq 0$. Multiplication is ordinary matrix multiplication and is *not* commutative. e is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} x & y \\ z & u \end{pmatrix}^{-1}$ is the usual matrix inverse.

Definition - a finite group with n elements is said to be *cyclic* if there is some element in it, call it a such that the elements $e, a, a^2, \ldots, a^{n-1}$ are all different. Clearly a^n must be e. We say that a generates the group.

Problem - Prove that for every $n = 1,\ 2, \ldots$ there exists a cyclic group with n elements.

Definition - The groups G_1 and G_2 are *isomorphic* if there exists a one-to-one function Φ with domain G_1 and range G_2 such that $\Phi(xy) = \Phi(x)\Phi(y)$ for all x and y. Φ is then called an *isomorphism* of G_1 and G_2. It is easy to prove the following (Try it):

1. If Φ is an isomorphism of G_1 and G_2 then Φ^{-1} is an isomorphism of G_2 and G_1.
2. $\Phi(e) = e$
3. $\Phi(x^{-1}) = (\Phi(x))^{-1}$

Example: The function $g(x) = \log(x)$ is an isomorphism of Example 2 at the top of the page and Example 1.

Problem - Prove that two finite cyclic groups with the same number of elements are isomorphic, i.e. to within isomorphism there is one and only one cyclic group of each finite order n.
The simplest way to define new cyclic groups is by using the direct product notion.

Def - Let G_1 and G_2 be any two groups. Let $G_1 \times G_2$ denote the set of all pairs x, y with $x \in G_1,\ y \in G_2$. Define multiplication by the rule $(x, y)(x', y') = (xx', yy')$. It is easy to show that this multiplication makes $G_1 \times G_2$ into a group. It is called the *direct product* of G_1 and G_2.

Problem (Easy) Show that the direct product of two cyclic groups of order 2 is not cyclic.

Problem (Slightly harder) Show that the direct product of two cyclic groups of the same order is not cyclic.

Problem - Show that *every* group with four elements is either isomorphic to a cyclic group of order four or is the direct product of two cyclic groups of order two.

Problem (not too easy) Let G be any finite group and let H be any "subgroup" i.e. any subset such that xy, x^{-1} and y^{-1} are in H whenever x

and y are, i.e. that H itself is a group. Show that the number of elements in H must be a divisor of the number of elements in G.
This theorem has a corllary that every group whose number of elements is a prime must be cyclic.

– – –

I fear I won't have time to finish what I had in mind before I see you Friday or Saturday. Maybe you need time to absorb the group theory anyway. Later in a final installment I will explain what harmonic analysis is, how it relates to group theory and how it helps to solve both partial differential equations and Diophantine equations of number theory thus bringing together the two 17th century sources. I also want to say a few words about the basic things one must learn before specializing.

With best wishes, Sincerely yours,
G. W. Mackey